CHAPTER

10

Function Operations

Throughout your mathematics courses, you have learned methods of interpreting a variety of functions. It is important to understand functional relationships between variables since they apply to the fields of engineering, business, physical sciences, and social sciences, to name a few.

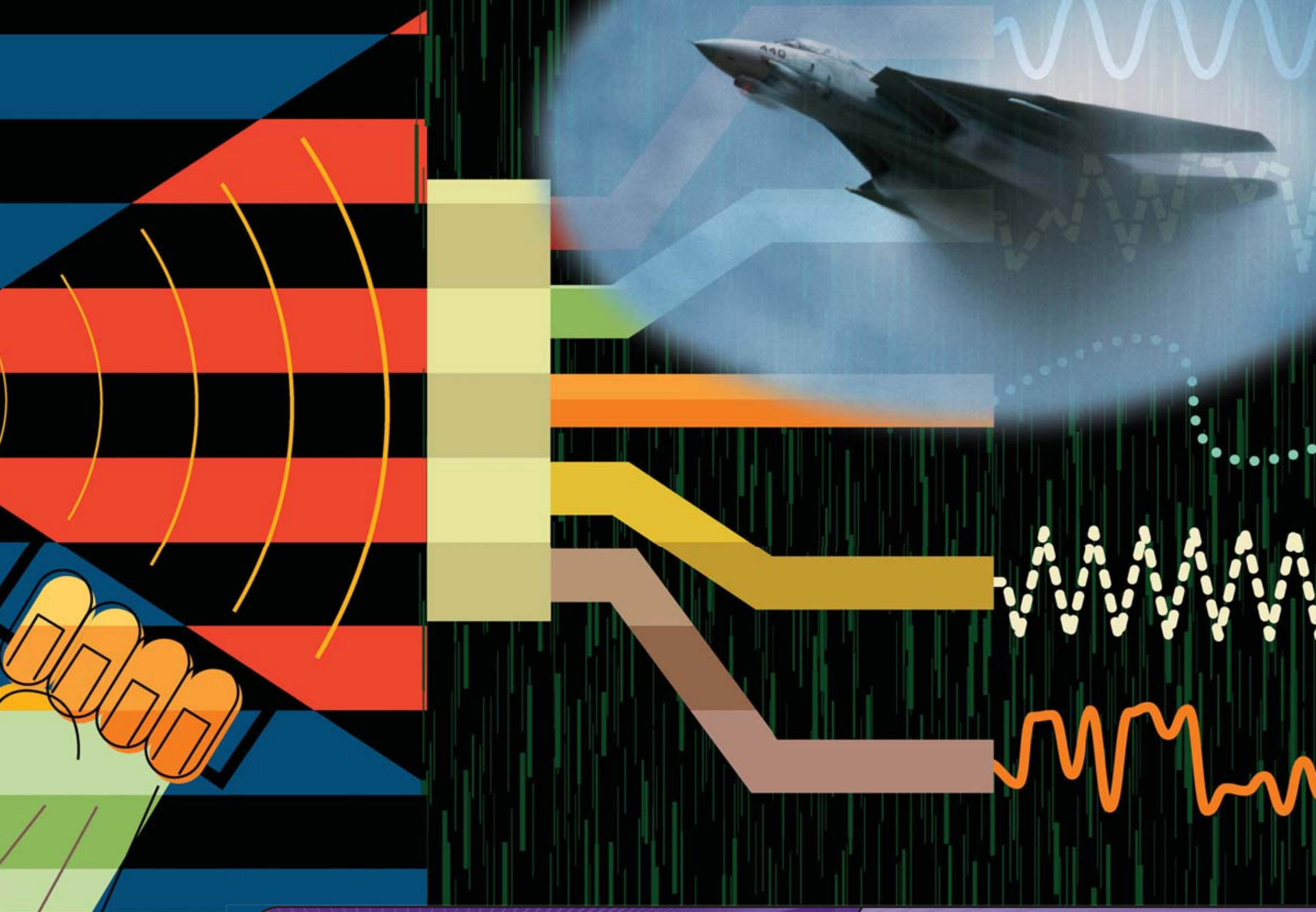
The relationships that exist between variables can be complex and can involve combining two or more functions. In this chapter, you will learn how to use various combinations of functions to model real-world phenomena.

Did You Know?

Wave interference occurs when two or more waves travel through the same medium at the same time. The net amplitude at each point of the resulting wave is the sum of the amplitudes of the individual waves. For example, waves interfere in wave pools and in noise-cancelling headphones.

Key Terms

composite function

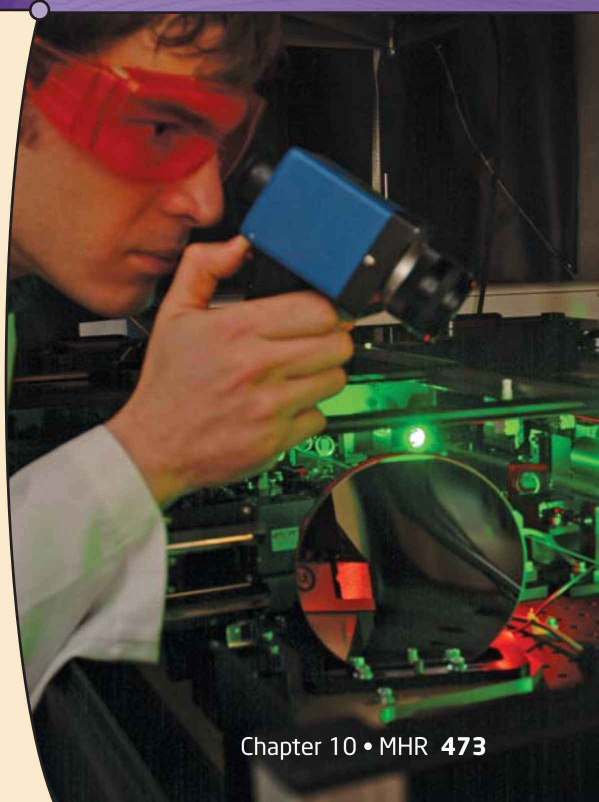


Career Link

In 2004, researchers from universities in British Columbia, Alberta, Ontario, and Québec, as well as from the National Research Council of Canada, began using the Advanced Laser Light Source (ALLS) to do fascinating experiments. The ALLS is a femtosecond (one quadrillionth (10^{-15}) of a second) multi-beam laser facility used in the dynamic investigation of matter in disciplines such as biology, medicine, chemistry, and physics. Universities such as the University of British Columbia offer students the chance to obtain advanced degrees leading to careers involving laser research.

Web Link

To learn more about a career involving laser research, go to www.mcgrawhill.ca/school/learningcentres and follow the links.



10.1

Sums and Differences of Functions

Focus on...

- sketching the graph of a function that is the sum or difference of two functions
- determining the domain and range of a function that is the sum or difference of two functions
- writing the equation of a function that is the sum or difference of two functions

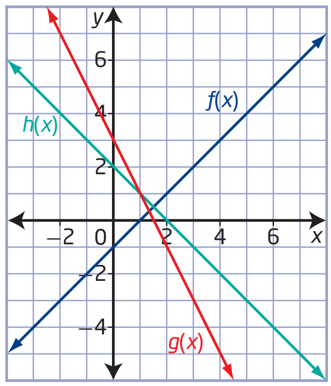
Physicists use ripple tanks to model wave motion. You know from previous work that sinusoidal functions can be used to model single wave functions. Waves are said to interfere with one another when they collide. Their collision can be modelled by the addition or subtraction of two sine waves.

Investigate Sums and Differences of Functions

Materials

- grid paper
- computer with spreadsheet software (optional)

1. Consider the graphs of the functions $f(x)$, $g(x)$, and $h(x)$.



a) Copy the table and use the graph of each function to complete the columns.

x	$f(x)$	$g(x)$	$h(x)$
−2			
−1			
0			
1			
2			
3			
4			

b) What do you notice about the relationship between each value of $h(x)$ and the corresponding values of $f(x)$ and $g(x)$?

2. a) Determine the equation, in slope-intercept form, of each function graphed in step 1.

   b) Using your equations, write the equation of a new function, $s(x)$, that represents the sum of the functions $f(x)$ and $g(x)$. Is this new function related to $h(x)$? Explain.

3. a) What are the domain and range of the functions $f(x)$ and $g(x)$?

   b) State the domain and range of $h(x)$. How do they relate to the domains and ranges of $f(x)$ and $g(x)$?

4. Add a fifth column to your table from step 1 using the heading $k(x) = f(x) - g(x)$, and fill in the values for the column.

5. a) Sketch the graphs of $f(x)$, $g(x)$, and $k(x)$ on the same set of coordinate axes.

   b) State the domain and range of $k(x)$. How do they relate to the domains and ranges of $f(x)$ and $g(x)$?

6. Using your equations from step 2, write the equation of a new function, $d(x)$, that represents the difference $g(x) - f(x)$. Is this new function related to $k(x)$? Explain.

Reflect and Respond

7. a) Choose two functions, $f(x)$ and $g(x)$, of your own. Sketch the graphs of $f(x)$ and $g(x)$ on the same set of coordinate axes.

   b) Explain how you can write the equation and produce the graph of the sum of $f(x)$ and $g(x)$.

   c) Explain how you can write the equation and produce the graph of the difference function, $g(x) - f(x)$.

8. Will your results of adding functions and subtracting functions apply to every type of function? Explain your reasoning.

Link the Ideas

You can form new functions by performing operations with functions.

To combine two functions, $f(x)$ and $g(x)$, add or subtract as follows:

Sum of Functions	*Difference of Functions*
$h(x) = f(x) + g(x)$	$h(x) = f(x) - g(x)$
$h(x) = (f + g)(x)$	$h(x) = (f - g)(x)$

Example 1

Determine the Sum of Two Functions

Consider the functions $f(x) = 2x + 1$ and $g(x) = x^2$.

a) Determine the equation of the function $h(x) = (f + g)(x)$.

b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

c) State the domain and range of $h(x)$.

d) Determine the values of $f(x)$, $g(x)$, and $h(x)$ when $x = 4$.

Solution

a) Add $f(x)$ and $g(x)$ to determine the equation of the function $h(x) = (f + g)(x)$.

$h(x) = (f + g)(x)$

$h(x) = f(x) + g(x)$

$h(x) = 2x + 1 + x^2$

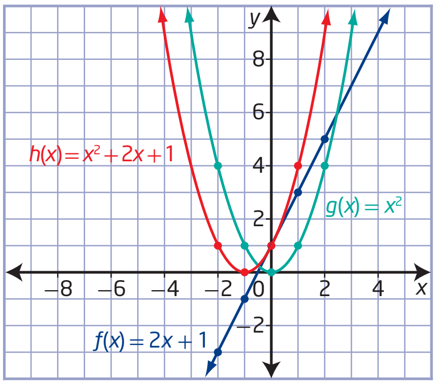
$h(x) = x^2 + 2x + 1$

Is the function $(f + g)(x)$ the same as $(g + f)(x)$? Will this always be true?

b) Method 1: Use Paper and Pencil

x	$f(x) = 2x + 1$	$g(x) = x^2$	$h(x) = x^2 + 2x + 1$
−2	−3	4	1
−1	−1	1	0
0	1	0	1
1	3	1	4
2	5	4	9

How could you use the values in the columns for $f(x)$ and $g(x)$ to determine the values in the column for $h(x)$?

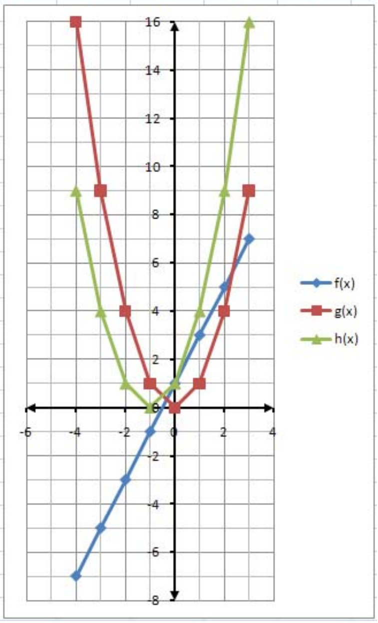


How are the y-coordinates of points on the graph of $h(x)$ related to those on the graphs of $f(x)$ and $g(x)$?

Method 2: Use a Spreadsheet

You can generate a table of values using a spreadsheet. From these values, you can create a graph.

x	f(x)	g(x)	h(x)
-4	-7	16	9
-3	-5	9	4
-2	-3	4	1
-1	-1	1	0
0	1	0	1
1	3	1	4
2	5	4	9
3	7	9	16



c) The function $f(x) = 2x + 1$ has domain $\{x \mid x \in R\}$.
The function $g(x) = x^2$ has domain $\{x \mid x \in R\}$.
The function $h(x) = (f + g)(x)$ has domain $\{x \mid x \in R\}$, which consists of all values that are in both the domain of $f(x)$ and the domain of $g(x)$.
The range of $h(x)$ is $\{y \mid y \geq 0, y \in R\}$.

d) Substitute $x = 4$ into $f(x)$, $g(x)$, and $h(x)$.

$f(x) = 2x + 1$	$g(x) = x^2$	$h(x) = x^2 + 2x + 1$
$f(4) = 2(4) + 1$	$g(4) = 4^2$	$h(4) = 4^2 + 2(4) + 1$
$f(4) = 8 + 1$	$g(4) = 16$	$h(4) = 16 + 8 + 1$
$f(4) = 9$		$h(4) = 25$

Your Turn

Consider the functions $f(x) = -4x - 3$ and $g(x) = 2x^2$.

a) Determine the equation of the function $h(x) = (f + g)(x)$.

b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

c) State the domain and range of $h(x)$.

Example 2

Determine the Difference of Two Functions

Consider the functions $f(x) = \sqrt{x - 1}$ and $g(x) = x - 2$.

a) Determine the equation of the function $h(x) = (f - g)(x)$.

b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

c) State the domain of $h(x)$.

d) Use the graph to approximate the range of $h(x)$.

Solution

a) Subtract $g(x)$ from $f(x)$ to determine the equation of the function $h(x) = (f - g)(x)$.

$h(x) = (f - g)(x)$
$h(x) = f(x) - g(x)$
$h(x) = \sqrt{x - 1} - (x - 2)$
$h(x) = \sqrt{x - 1} - x + 2$

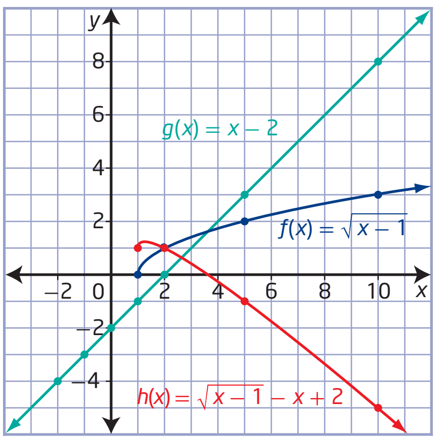
b) **Method 1: Use Paper and Pencil**

For the function $f(x) = \sqrt{x - 1}$, the value of the radicand must be greater than or equal to zero: $x - 1 \geq 0$ or $x \geq 1$.

x	$f(x) = \sqrt{x - 1}$	$g(x) = x - 2$	$h(x) = \sqrt{x - 1} - x + 2$
−2	undefined	−4	undefined
−1	undefined	−3	undefined
0	undefined	−2	undefined
1	0	−1	1
2	1	0	1
5	2	3	−1
10	3	8	−5

Why is the function $h(x)$ undefined when $x < 1$?

How could you use the values in the columns for $f(x)$ and $g(x)$ to determine the values in the column for $h(x)$?

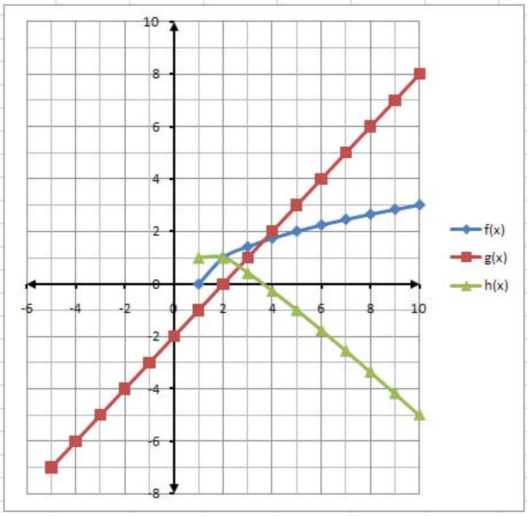


How could you use the y-coordinates of points on the graphs of $f(x)$ and $g(x)$ to create the graph of $h(x)$?

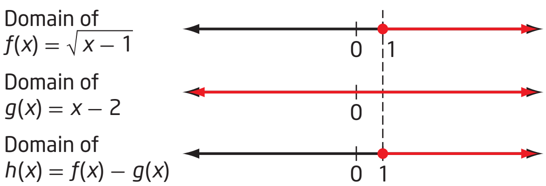
Method 2: Use a Spreadsheet

You can generate a table of values using a spreadsheet. From these values, you can create a graph.

	A	B	C	D
1	x	f(x)	g(x)	h(x)
2	-5	#NUM!	-7	#NUM!
3	-4	#NUM!	-6	#NUM!
4	-3	#NUM!	-5	#NUM!
5	-2	#NUM!	-4	#NUM!
6	-1	#NUM!	-3	#NUM!
7	0	#NUM!	-2	#NUM!
8	1	0	-1	1
9	2	1	0	1
10	3	1.414214	1	0.414214
11	4	1.732051	2	-0.26795
12	5	2	3	-1
13	6	2.236068	4	-1.76393
14	7	2.44949	5	-2.55051
15	8	2.645751	6	-3.35425
16	9	2.828427	7	-4.17157
17	10	3	8	-5



c) The function $f(x) = \sqrt{x - 1}$ has domain $\{x \mid x \geq 1, x \in \mathrm{R}\}$.
The function $g(x) = x - 2$ has domain $\{x \mid x \in \mathrm{R}\}$.
The function $h(x) = (f - g)(x)$ has domain $\{x \mid x \geq 1, x \in \mathrm{R}\}$, which consists of all values that are in both the domain of $f(x)$ and the domain of $g(x)$.



What values of x belong to the domains of both $f(x)$ and $g(x)$?

d) From the graph, the range of $h(x)$ appears to be approximately $\{y \mid y \leq 1.2, y \in \mathrm{R}\}$.

How can you use a graphing calculator to verify the range?

Your Turn

Consider the functions $f(x) = |x|$ and $g(x) = x - 5$.

a) Determine the equation of the function $h(x) = (f - g)(x)$.

b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

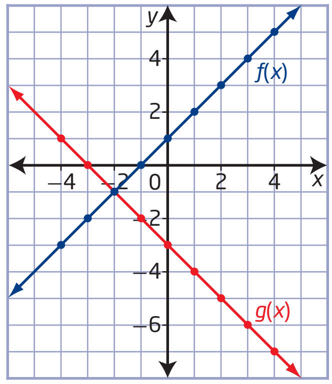
c) State the domain and range of $h(x)$.

d) Is $(f - g)(x)$ equal to $(g - f)(x)$? If not, what are the similarities and differences?

Example 3

Determine a Combined Function From Graphs

Sketch the graph of $h(x) = (f + g)(x)$ given the graphs of $f(x)$ and $g(x)$.

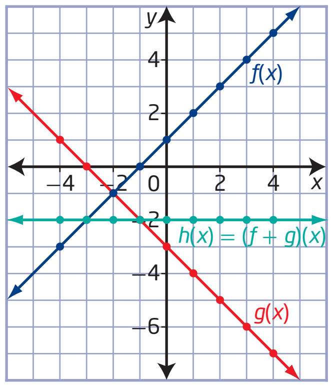


Solution

Method 1: Add the *y*-Coordinates of Corresponding Points
Create a table of values from the graphs of $f(x)$ and $g(x)$.

Add the y-coordinates at each point to determine points on the graph of $h(x) = (f + g)(x)$. Plot these points and draw the graph of $h(x) = (f + g)(x)$.

x	$f(x)$	$g(x)$	$h(x) = (f + g)(x)$
−4	−3	1	$-3 + 1 = -2$
−3	−2	0	$-2 + 0 = -2$
−2	−1	−1	$-1 + (-1) = -2$
−1	0	−2	$0 + (-2) = -2$
0	1	−3	$1 + (-3) = -2$
1	2	−4	$2 + (-4) = -2$
2	3	−5	$3 + (-5) = -2$
3	4	−6	$4 + (-6) = -2$
4	5	−7	$5 + (-7) = -2$



The result is a line with a slope of 0 and a y-intercept of -2. Therefore, $h(x) = -2$.

Method 2: Determine the Equations
To determine $h(x)$, you could first determine the equations of $f(x)$ and $g(x)$.

For the graph of $f(x)$, the y-intercept is 1 and the slope is 1. So, the equation is $f(x) = x + 1$.

For $g(x)$, the equation is $g(x) = -x - 3$.

What are the slope and *y*-intercept of this line?

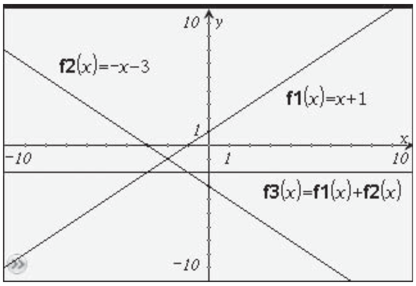
Determine the equation of $h(x)$ algebraically.
$h(x) = (f + g)(x)$
$h(x) = x + 1 + (-x - 3)$
$h(x) = -2$

The graph of $h(x) = -2$ is a horizontal line.

Did You Know?

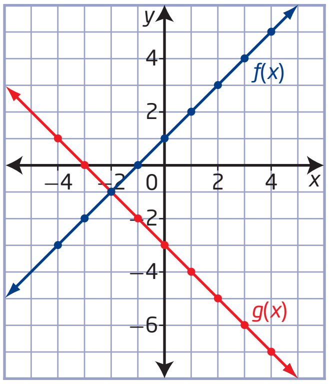
When the order of operands can be reversed and still produce the same result, the operation is said to be commutative. For example, $(f + g)(x) = (g + f)(x)$

You can verify your answer by graphing $f(x) = x + 1$, $g(x) = -x - 3$, and $h(x) = f(x) + g(x)$ using a graphing calculator.



Your Turn

Sketch the graph of $m(x) = (f - g)(x)$ given the graphs of $f(x)$ and $g(x)$.



Example 4

Application of the Difference of Two Functions

Reach for the Top is an academic challenge program offered to students across Canada. Suppose the cost of T-shirts for the program includes \$125 in fixed costs and \$7.50 per T-shirt. The shirts are sold for \$12.00 each.

a) Write an equation to represent
- the total cost, C, as a function of the number, n, of T-shirts produced
- the revenue, R, as a function of the number, n, of T-shirts sold

b) Graph the total cost and revenue functions on the same set of axes. What does the point of intersection represent?

c) Profit, P, is the difference between revenue and cost. Write a function representing P in terms of n.

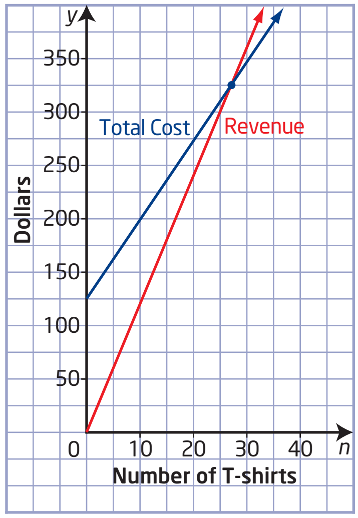
d) Identify the domain of the total cost, revenue, and profit functions in the context of this problem.

Solution

a) The total cost of producing the T-shirts can be represented by the function $C(n) = 7.5n + 125$.
The revenue can be represented by the function $R(n) = 12n$.

b) Graph the functions.

The point of intersection represents the point at which the total cost equals the revenue, or the break-even point. Any further sales of T-shirts will result in profit.



c) Profit can be represented by a combined function:
$P(n) = R(n) - C(n)$
$P(n) = 12n - (7.5n + 125)$
$P(n) = 4.5n - 125$

d) The domain of $C(n) = 7.5n + 125$ is $\{n \mid n \geq 0, n \in W\}$.

The domain of $R(n) = 12n$ is $\{n \mid n \geq 0, n \in W\}$.

The domain of $P(n) = 4.5n - 125$ is $\{n \mid n \geq 0, n \in W\}$.

Your Turn

Math Kangaroo is an international mathematics competition that is held in over 40 countries, including Canada. Suppose the cost of preparing booklets for the Canadian version of the contest includes \$675 in fixed costs and \$3.50 per booklet. The booklets are sold for \$30 each.

a) Write an equation to represent
- the total cost, C, as a function of the number, n, of booklets produced
- the revenue, R, as a function of the number, n, of booklets sold
- the profit, P, the difference between revenue and total cost

b) Graph the total cost, revenue, and profit functions on the same set of axes. How many booklets must be sold to make a profit?

c) Identify the domain of the total cost, revenue, and profit functions in the context of this problem.

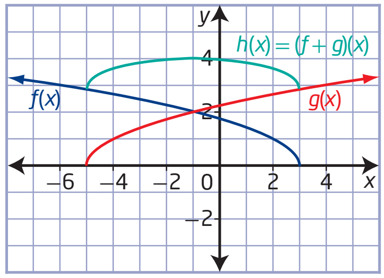
Did You Know?

Math Kangaroo or *Kangourou sans frontières* originated in France in 1991. The first Canadian edition of the competition was held in 2001.

Key Ideas

- You can add two functions, $f(x)$ and $g(x)$, to form the combined function $h(x) = (f + g)(x)$.
- You can subtract two functions, $f(x)$ and $g(x)$, to form the combined function $h(x) = (f - g)(x)$.
- The domain of the combined function formed by the sum or difference of two functions is the domain common to the individual functions. For example,

  Domain of $f(x)$: $\{x \mid x \leq 3, x \in \mathrm{R}\}$
  Domain of $g(x)$: $\{x \mid x \geq -5, x \in \mathrm{R}\}$
  Domain of $h(x)$: $\{x \mid -5 \leq x \leq 3, x \in \mathrm{R}\}$



- The range of a combined function can be determined using its graph.
- To sketch the graph of a sum or difference of two functions given their graphs, add or subtract the y-coordinates at each point.

Check Your Understanding

Practise

1. For each pair of functions, determine $h(x) = f(x) + g(x)$.

a) $f(x) = |x - 3|$ and $g(x) = 4$

b) $f(x) = 3x - 5$ and $g(x) = -x + 2$

c) $f(x) = x^2 + 2x$ and $g(x) = x^2 + x + 2$

d) $f(x) = -x - 5$ and $g(x) = (x + 3)^2$

2. For each pair of functions, determine $h(x) = f(x) - g(x)$.

a) $f(x) = 6x$ and $g(x) = x - 2$

b) $f(x) = -3x + 7$ and $g(x) = 3x^2 + x - 2$

c) $f(x) = 6 - x$ and $g(x) = (x + 1)^2 - 7$

d) $f(x) = \cos x$ and $g(x) = 4$

3. Consider $f(x) = -6x + 1$ and $g(x) = x^2$.

a) Determine $h(x) = f(x) + g(x)$ and find $h(2)$.

b) Determine $m(x) = f(x) - g(x)$ and find $m(1)$.

c) Determine $p(x) = g(x) - f(x)$ and find $p(1)$.

4. Given $f(x) = 3x^2 + 2$, $g(x) = \sqrt{x + 4}$, and $h(x) = 4x - 2$, determine each combined function and state its domain.

a) $y = (f + g)(x)$ **b)** $y = (h - g)(x)$

c) $y = (g - h)(x)$ **d)** $y = (f + h)(x)$

5. Let $f(x) = 2^x$ and $g(x) = 1$. Graph each of the following, stating its domain and range.

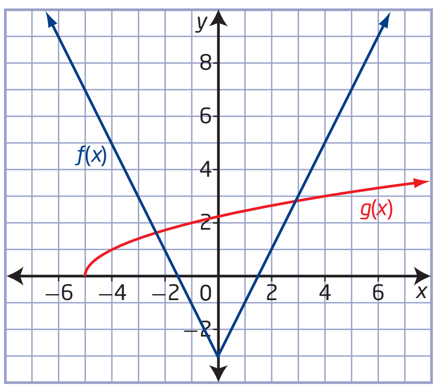
a) $y = (f + g)(x)$ **b)** $y = (f - g)(x)$

c) $y = (g - f)(x)$

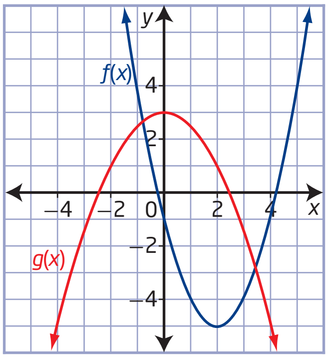
6. Use the graphs of $f(x)$ and $g(x)$ to evaluate the following.

a) $(f + g)(4)$ **b)** $(f + g)(-4)$

c) $(f + g)(-5)$ **d)** $(f + g)(-6)$



7. Use the graphs of $f(x)$ and $g(x)$ to determine which graph matches each combined function.

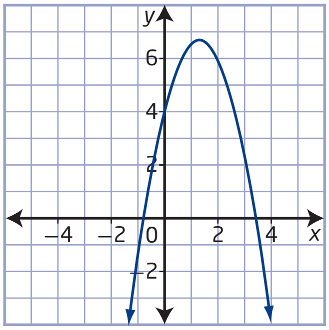


a) $y = (f + g)(x)$

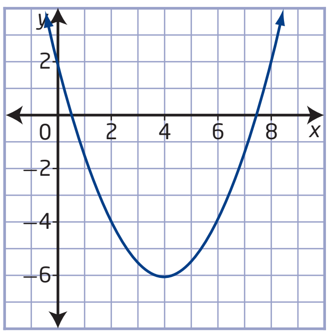
b) $y = (f - g)(x)$

c) $y = (g - f)(x)$

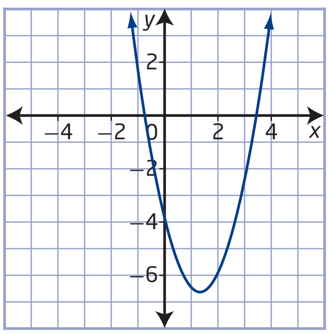
A



B

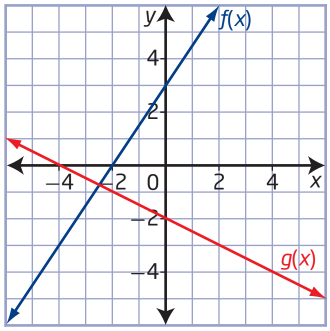


C

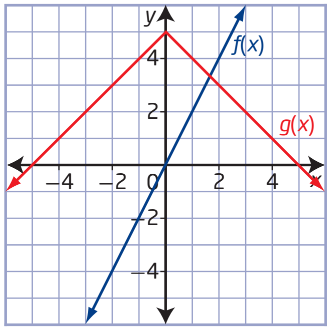


8. Copy each graph. Add the sketch of the graph of each combined function to the same set of axes.

a)



b)



i) $y = (f + g)(x)$

ii) $y = (f - g)(x)$

iii) $y = (g - f)(x)$

Apply

9. Given $f(x) = 3x^2 + 2$, $g(x) = 4x$, and $h(x) = 7x - 1$, determine each combined function.

a) $y = f(x) + g(x) + h(x)$

b) $y = f(x) + g(x) - h(x)$

c) $y = f(x) - g(x) + h(x)$

d) $y = f(x) - g(x) - h(x)$

10. If $h(x) = (f + g)(x)$ and $f(x) = 5x + 2$, determine $g(x)$.

a) $h(x) = x^2 + 5x + 2$

b) $h(x) = \sqrt{x + 7} + 5x + 2$

c) $h(x) = 2x + 3$

d) $h(x) = 3x^2 + 4x - 2$

11. If $h(x) = (f - g)(x)$ and $f(x) = 5x + 2$, determine $g(x)$.

a) $h(x) = -x^2 + 5x + 3$

b) $h(x) = \sqrt{x - 4} + 5x + 2$

c) $h(x) = -3x + 11$

d) $h(x) = -2x^2 + 16x + 8$

12. An eco-friendly company produces a water bottle waist pack from recycled plastic. The supply, S, in hundreds of waist packs, is a function of the price, p, in dollars, and is modelled by the function $S(p) = p + 4$. The demand, D, for the waist packs is modelled by $D(p) = -0.1(p + 8)(p - 10)$.

a) Graph these functions on the same set of axes. What do the points of intersection represent? Should both points be considered? Explain.

b) Graph the function $y = S(p) - D(p)$. Explain what it models.

13. The daily costs for a hamburger vendor are \$135 per day plus \$1.25 per hamburger sold. He sells each burger for \$3.50, and the maximum number of hamburgers he can sell in a day is 300.

a) Write equations to represent the total cost, C, and the total revenue, R, as functions of the number, n, of hamburgers sold.

b) Graph $C(n)$ and $R(n)$ on the same set of axes.

c) The break-even point is where $C(n) = R(n)$. Identify this point.

d) Develop an algebraic and a graphical model for the profit function.

e) What is the maximum daily profit the vendor can earn?

14. Two waves are generated in a ripple tank. Suppose the height, in centimetres, above the surface of the water, of the waves can be modelled by $f(x) = \sin x$ and $g(x) = 3 \sin x$, where x is in radians.

a) Graph $f(x)$ and $g(x)$ on the same set of coordinate axes.

b) Use your graph to sketch the graph of $h(x) = (f + g)(x)$.

c) What is the maximum height of the resultant wave?

15. Automobile mufflers are designed to reduce exhaust noise in part by applying wave interference. The resonating chamber of a muffler contains a specific volume of air and has a specific length that is calculated to produce a wave that cancels out a certain frequency of sound. Suppose the engine noise can be modelled by $E(t) = 10 \sin 480\pi t$ and the resonating chamber produces a wave modelled by $R(t) = 8 \sin 480\pi(t - 0.002)$, where t is the time, in seconds.

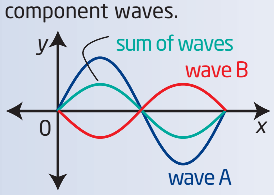
a) Graph $E(t)$ and $R(t)$ using technology for a time period of 0.02 s.

b) Describe the general relationship between the locations of the maximum and minimum values of the two functions. Will this result in destructive interference or constructive interference?

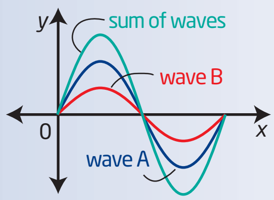
c) Graph $E(t) + R(t)$.

Did You Know?

Destructive interference occurs when the sum of two waves has a lesser amplitude than the component waves.



Constructive interference occurs when the sum of two waves has a greater amplitude than the component waves.



16. An alternating current–direct current (AC-DC) voltage signal is made up of the following two components, each measured in volts (V): $V_{AC}(t) = 10 \sin t$ and $V_{DC}(t) = 15$.

a) Sketch the graphs of these two functions on the same set of axes. Work in radians.

b) Graph the combined function $V_{AC}(t) + V_{DC}(t)$.

c) Identify the domain and range of $V_{AC}(t) + V_{DC}(t)$.

d) Use the range of the combined function to determine the following values of this voltage signal.

i) minimum

ii) maximum

17. During a race in the Sportsman category of drag racing, it is common for cars with different performance potentials to race against each other while using a handicap system. Suppose the distance, d_1, in metres, that the faster car travels is given by $d_1(t) = 10t^2$, where t is the time, in seconds, after the driver starts. The distance, d_2, in metres, that the slower car travels is given by $d_2(t) = 5(t + 2)^2$, where t is the time, in seconds, after the driver of the faster car starts. Write a function, $h(t)$, that gives the relative distance between the cars over time.

Did You Know?

The Saskatchewan International Raceway is the oldest drag strip in Western Canada. It was built in 1966 and is located outside of Saskatoon.

18. a) Use technology to graph $f(x) = \sin x$ and $g(x) = x$, where x is in radians, on the same graph.

b) Predict the shape of $h(x) = f(x) + g(x)$. Verify your prediction using graphing technology.

Extend

19. A skier is skiing through a series of moguls down a course that is 200 m in length at a constant speed of 1 m/s. The constant slope of the hill is −1.

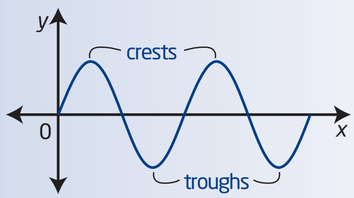
a) Write a function representing the skier's distance, d, from the base of the hill versus time, t, in seconds (neglecting the effects of the moguls).

b) If the height, m, of the skier through the moguls, ignoring the slope of the hill, is $m(t) = 0.75 \sin 1.26t$, write a function that represents the skier's actual path of height versus time.

c) Graph the function in part a) and the two functions in part b) on the same set of axes.

Did You Know?

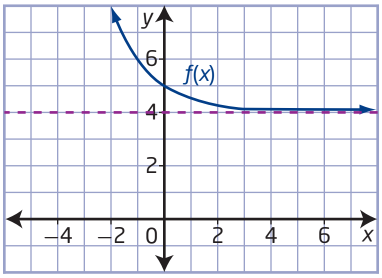
Moguls are representative wave models. The crests are the high points of a mogul and the troughs are the low points.



20. An *even function* satisfies the property $f(-x) = f(x)$ for all x in the domain of $f(x)$. An *odd function* satisfies the property $f(-x) = -f(x)$ for all x in the domain of $f(x)$.

Devise and test an algebraic method to determine if the sum of two functions is even, odd, or neither. Show by example how your method works. Use at least three of the functions you have studied: absolute value, radical, polynomial, trigonometric, exponential, logarithmic, and rational.

21. The graph shows either the sum or the difference of two functions. Identify the types of functions that were combined. Verify your thinking using technology.



22. Consider $f(x) = x^2 - 9$ and $g(x) = \frac{1}{x}$.

a) State the domain and range of each function.

b) Determine $h(x) = f(x) + g(x)$.

c) How do the domain and range of each function compare to the domain and range of $h(x)$?

Create Connections

C1 **a)** Is $f(x) + g(x) = g(x) + f(x)$ true for all functions? Justify your answer.

b) Is $(f - g)(x) = (g - f)(x)$ true for all functions? Justify your answer.

C2 Let $y_1 = x^3$ and $y_2 = 4$. Use graphs, numbers, and words to determine

a) the function $y_3 = y_1 + y_2$

b) the domain and range of y_3

C3 **MINI LAB**

Materials

- grid paper or graphing technology

Model the path of a bungee jumper. The table gives the height versus time data of a bungee jumper. Heights are referenced to the rest position of the bungee jumper, which is well above ground level.

Time (s)	Height (m)	Time (s)	Height (m)	Time (s)	Height (m)
0	100	13	−11	26	−39
1	90	14	11	27	−39
2	72	15	30	28	−35
3	45	16	44	29	−27
4	14	17	53	30	−16
5	−15	18	54	31	−4
6	−41	19	48	32	6
7	−61	20	37	33	17
8	−71	21	23	34	24
9	−73	22	6	35	28
10	−66	23	−8	36	29
11	−52	24	−23	37	26
12	−32	25	−33		

Step 1 Create a graph of height versus time. How does the graph exhibit sinusoidal features?

Step 2 Describe how the graph exhibits exponential features.

Step 3 Construct a cosine function that has the same period (wavelength) as the graph in step 1.

Step 4 Construct an exponential function to model the decay in amplitude of the graph in step 1.

Step 5 Construct a combined function to model the height-time relationship of the bungee jumper.

Step 6 How far will the bungee jumper be above the rest position at his fourth crest?

10.2

Products and Quotients of Functions

Focus on...

- sketching the graph of a function that is the product or quotient of two functions
- determining the domain and range of a function that is the product or quotient of two functions
- writing the equation of a function that is the product or quotient of two functions

Winnipeg Jets celebrate scoring in game against Montreal at MTS Centre Winnipeg on October 9, 2011

You have explored how functions can be combined through addition and subtraction. When combining functions using products or quotients, you will use techniques similar to those you learned when multiplying and dividing rational expressions, including the identification of non-permissible values.

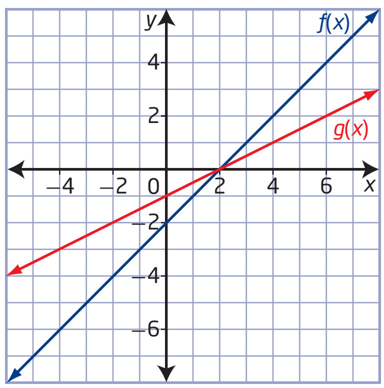
You can use products and quotients of functions to solve problems related to populations, revenues at a sports venue, and the movement of a pendulum on a clock, to name a few examples.

Investigate Products and Quotients of Functions

Materials

- grid paper

1. Consider the graphs of the functions $f(x)$ and $g(x)$.



a) Copy the table and use the graph of each function to complete the columns for $f(x)$ and $g(x)$. In the last column, enter the product of the values of $f(x)$ and $g(x)$.

x	$f(x)$	$g(x)$	$p(x)$
−4			
−2			
0			
2			
4			
6			

b) Predict the shape of the graph of $p(x)$. Then, copy the graphs of $f(x)$ and $g(x)$ and sketch the graph of $p(x)$ on the same set of axes.

c) How are the x-intercepts of the graph of $p(x)$ related to those of the graphs of $f(x)$ and $g(x)$?

2. a) Determine the equations of the functions $f(x)$ and $g(x)$.

b) How can you use the equations of $f(x)$ and $g(x)$ to write the equation of the function $p(x)$? Write the equation of $p(x)$ and verify that it matches the graph.

3. a) What are the domain and range of the functions $f(x)$, $g(x)$, and $p(x)$?

b) Is the relationship between the domains of $f(x)$, $g(x)$, and $p(x)$ the same as it was for addition and subtraction of functions? Explain.

4. a) Add a fifth column to your table from step 1 using the heading $q(x)$. Fill in the column with the quotient of the values of $f(x)$ and $g(x)$.

b) Predict the shape of the graph of $q(x)$. Then, copy the graphs of $f(x)$ and $g(x)$ and sketch the graph of $q(x)$ on the same set of axes.

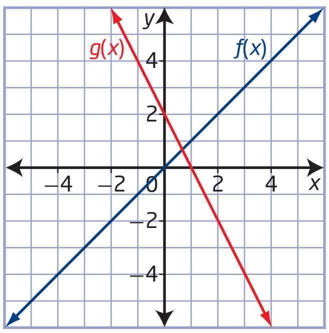
c) How are the x-intercepts of the graphs of $f(x)$ and $g(x)$ related to the values of $q(x)$?

5. a) How can you use the equations from step 2a) to write the equation of the function $q(x)$? Write the equation of $q(x)$ and verify that it matches the graph.

b) State the domain and range of $q(x)$. Is the relationship between the domains of $f(x)$, $g(x)$, and $q(x)$ the same as it was for addition and subtraction of functions? Explain.

Reflect and Respond

6. Consider the graphs of $f(x) = x$ and $g(x) = -2x + 2$.



a) Explain how you can write the equation and produce the graph of the product of $f(x)$ and $g(x)$.

b) Explain how you can write the equation and produce the graph of the quotient of $f(x)$ and $g(x)$.

c) What must you consider when determining the domain of a product of functions or the domain of a quotient of functions?

Link the Ideas

Did You Know?

Multiplication can be shown using a centred dot. For example, $2 \times 5 = 2 \cdot 5$

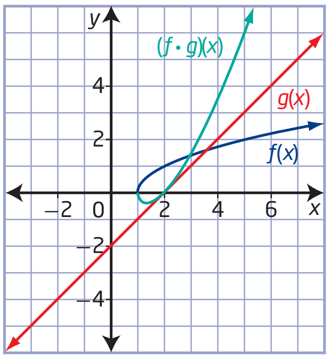
To combine two functions, $f(x)$ and $g(x)$, multiply or divide as follows:

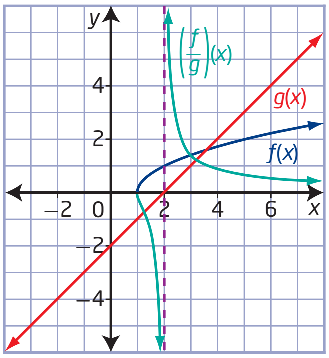
Product of Functions	*Quotient of Functions*
$h(x) = f(x)g(x)$	$h(x) = \frac{f(x)}{g(x)}$
$h(x) = (f \cdot g)(x)$	$h(x) = \left(\frac{f}{g}\right)(x)$

The domain of a product of functions is the domain common to the original functions. However, the domain of a quotient of functions must take into consideration that division by zero is undefined. The domain of a quotient, $h(x) = \frac{f(x)}{g(x)}$, is further restricted for values of x where $g(x) = 0$.

Consider $f(x) = \sqrt{x - 1}$ and $g(x) = x - 2$.

The domain of $f(x)$ is $\{x \mid x \geq 1, x \in \mathrm{R}\}$, and the domain of $g(x)$ is $\{x \mid x \in \mathrm{R}\}$. So, the domain of $(f \cdot g)(x)$ is $\{x \mid x \geq 1, x \in \mathrm{R}\}$, while the domain of $\left(\frac{f}{g}\right)(x)$ is $\{x \mid x \geq 1, x \neq 2, x \in \mathrm{R}\}$





Example 1

Determine the Product of Functions

Given $f(x) = (x + 2)^2 - 5$ and $g(x) = 3x - 4$, determine $h(x) = (f \cdot g)(x)$. State the domain and range of $h(x)$.

Solution

To determine $h(x) = (f \cdot g)(x)$, multiply the two functions.

$h(x) = (f \cdot g)(x)$
$h(x) = f(x)g(x)$
$h(x) = ((x + 2)^2 - 5)(3x - 4)$
$h(x) = (x^2 + 4x - 1)(3x - 4)$
$h(x) = 3x^3 - 4x^2 + 12x^2 - 16x - 3x + 4$
$h(x) = 3x^3 + 8x^2 - 19x + 4$

How can you tell from the original functions that the product is a cubic function?

The function $f(x) = (x + 2)^2 - 5$ is quadratic with domain $\{x \mid x \in \mathrm{R}\}$.

The function $g(x) = 3x - 4$ is linear with domain $\{x \mid x \in \mathrm{R}\}$.

The domain of $h(x) = (f \cdot g)(x)$ consists of all values that are in both the domain of $f(x)$ and the domain of $g(x)$.

Therefore, the cubic function $h(x) = 3x^3 + 8x^2 - 19x + 4$ has domain $\{x \mid x \in \mathrm{R}\}$ and range $\{y \mid y \in \mathrm{R}\}$.

Your Turn

Given $f(x) = x^2$ and $g(x) = \sqrt{4x - 5}$, determine $h(x) = f(x)g(x)$. State the domain and range of $h(x)$.

Example 2

Determine the Quotient of Functions

Consider the functions $f(x) = x^2 + x - 6$ and $g(x) = 2x + 6$.

a) Determine the equation of the function $h(x) = \left(\frac{g}{f}\right)(x)$.

b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

c) State the domain and range of $h(x)$.

Solution

a) To determine $h(x) = \left(\frac{g}{f}\right)(x)$, divide the two functions.

$$h(x) = \left(\frac{g}{f}\right)(x)$$

$$h(x) = \frac{g(x)}{f(x)}$$

$$h(x) = \frac{2x + 6}{x^2 + x - 6}$$

$$h(x) = \frac{2(x + 3)}{(x + 3)(x - 2)} \qquad \text{Factor.}$$

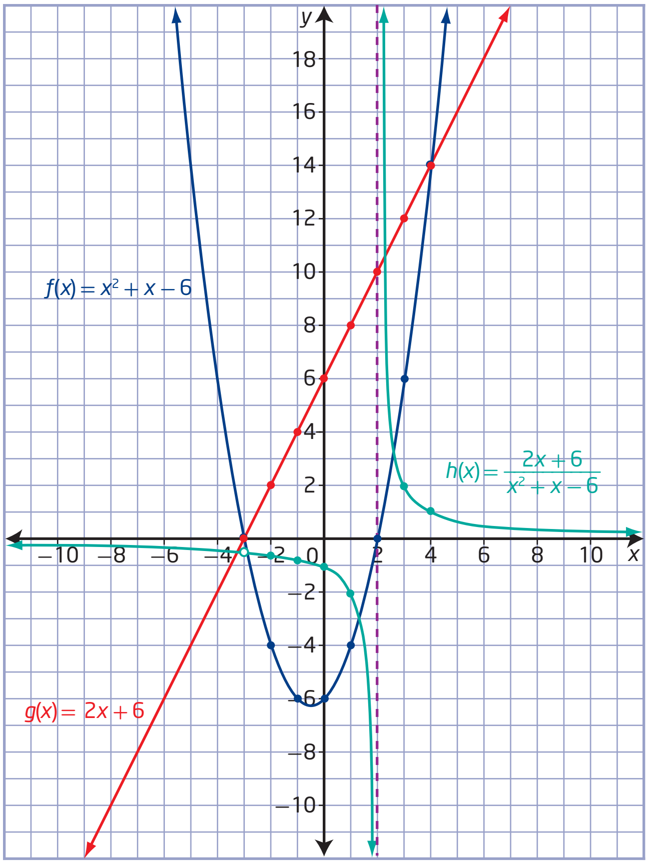
$$h(x) = \frac{2\overset{1}{\cancel{(x + 3)}}}{\underset{1}{\cancel{(x + 3)}}(x - 2)}$$

$$h(x) = \frac{2}{x - 2}, \; x \neq -3, 2 \qquad \text{Identify any non-permissible values.}$$

b) Method 1: Use Paper and Pencil

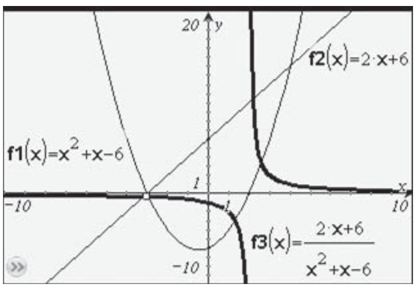
How could you use the values in the columns for $f(x)$ and $g(x)$ to determine the values in the column for $h(x)$?

x	$f(x) = x^2 + x - 6$	$g(x) = 2x + 6$	$h(x) = \frac{2}{x - 2}, x \neq -3, 2$
−3	0	0	does not exist
−2	−4	2	$-\frac{1}{2}$
−1	−6	4	$-\frac{2}{3}$
0	−6	6	−1
1	−4	8	−2
2	0	10	undefined
3	6	12	2
4	14	14	1



How are the y-coordinates of points on the graph of $h(x)$ related to those on the graphs of $f(x)$ and $g(x)$?

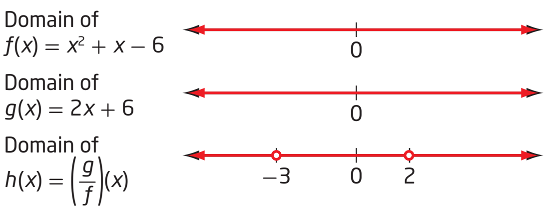
Method 2: Use a Graphing Calculator



c) The function $f(x) = x^2 + x - 6$ is quadratic with domain $\{x \mid x \in \mathrm{R}\}$. The function $g(x) = 2x + 6$ is linear with domain $\{x \mid x \in \mathrm{R}\}$. The domain of $h(x) = \left(\frac{g}{f}\right)(x)$ consists of all values that are in both the domain of $f(x)$ and the domain of $g(x)$, excluding values of x where $f(x) = 0$.

Since the function $h(x)$ does not exist at $\left(-3, -\frac{2}{5}\right)$ and is undefined at $x = 2$, the domain is $\{x \mid x \neq -3, x \neq 2, x \in \mathrm{R}\}$. This is shown in the graph by the point of discontinuity at $\left(-3, -\frac{2}{5}\right)$ and the vertical asymptote that appears at $x = 2$.

How do you know there is a point of discontinuity and an asymptote?



The range of $h(x)$ is $\left\{y \mid y \neq 0, -\frac{2}{5}, y \in \mathrm{R}\right\}$.

Your Turn

Let $f(x) = x + 2$ and $g(x) = x^2 + 9x + 14$.

a) Determine the equation of the function $h(x) = \left(\frac{f}{g}\right)(x)$.

b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

c) State the domain and range of $h(x)$.

Example 3

Application of Products and Quotients of Functions

A local hockey team owner would like to boost fan support at the games. He decides to reduce the ticket prices, T, in dollars, according to the function $T(g) = 10 - 0.1g$, where g represents the game number.

To further increase fan support, he decides to randomly give away noisemakers. The number, N, in hundreds, of noisemakers can be modelled by the function $N(g) = 2 - 0.05g$. The community fan base is small but the owner notices that since the incentives were put in place, attendance, A, in hundreds, can be modelled by $A(g) = 8 + 0.2g$.

a) Determine $r(g) = T(g)A(g)$ algebraically and explain what it represents.

b) Use the graph of $r(g) = T(g)A(g)$ to determine whether the owner increases or decreases revenue from the ticket sales with the changes made to draw in new fans.

c) Develop an algebraic and a graphical model for $p(g) = \frac{N(g)}{A(g)}$. Explain what it means. What is the chance of receiving a free noisemaker for a fan attending game 4?

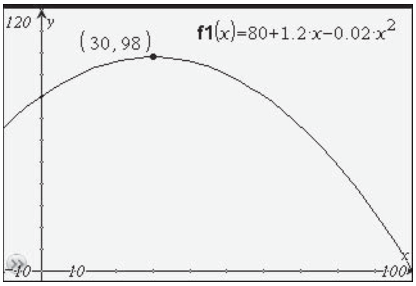
Solution

a) Multiply $T(g)$ by $A(g)$ to produce $r(g) = T(g)A(g)$.

$r(g) = T(g)A(g)$
$r(g) = (10 - 0.1g)(8 + 0.2g)$
$r(g) = 80 + 2g - 0.8g - 0.02g^2$
$r(g) = 80 + 1.2g - 0.02g^2$

This new function multiplies the ticket price by the attendance. Therefore, $r(g) = T(g)A(g)$ represents the revenue from ticket sales, in hundreds of dollars.

b) Enter $r(g) = 80 + 1.2g - 0.02g^2$ on a graphing calculator. Then, use the trace feature or the table feature to view increasing and decreasing values.



The graph of $r(g) = 80 + 1.2g - 0.02g^2$ is a parabola that continues to increase until game 30, at which time it begins to decrease.

The owner will increase revenue up to a maximum at 30 games, after which revenue will decrease.

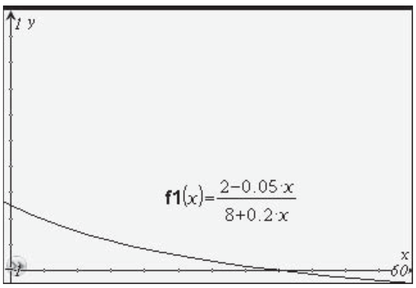
c) To determine the quotient, divide $N(g)$ by $A(g)$.

$$p(g) = \frac{N(g)}{A(g)}$$

$$p(g) = \frac{2 - 0.05g}{8 + 0.2g}$$

What is the non-permissible value of g? How does it affect this situation?

Use graphing technology to graph the new function.



Which window settings would you use in this situation?

This combined function represents the number of free noisemakers that will be randomly handed out divided by the number of fans attending. This function represents the probability that a fan will receive a free noisemaker as a function of the game number.

To determine the probability of receiving a free noisemaker at game 4, evaluate $p(g) = \frac{2 - 0.05g}{8 + 0.2g}$ for $g = 4$.

$$p(g) = \frac{2 - 0.05g}{8 + 0.2g}$$
$$p(4) = \frac{2 - 0.05(4)}{8 + 0.2(4)}$$
$$p(4) = \frac{1.8}{8.8}$$
$$p(4) = 0.2045\ldots$$

There is approximately a 0.20 or 20% chance of receiving a free noisemaker.

Did You Know?

The probability that an event will occur is the total number of favourable outcomes divided by the total number of possible outcomes.

Your Turn

A pendulum is released and allowed to swing back and forth. The periodic nature of the motion is described as $p(t) = 10 \cos 2t$, where p is the horizontal displacement, in centimetres, from the pendulum's resting position as a function of time, t, in seconds. The decay of the amplitude is given by $q(t) = 0.95^t$.

a) Write the combined function that is the product of the two components. Explain what the product represents.

b) Graph the combined function. Describe its characteristics and explain how the graph models the motion of the pendulum.

Did You Know?

In 1851, Jean Foucault demonstrated that the Earth rotates by using a long pendulum that swung in the same plane while the Earth rotated beneath it.

Key Ideas

- The combined function $h(x) = (f \cdot g)(x)$ represents the product of two functions, $f(x)$ and $g(x)$.
- The combined function $h(x) = \left(\frac{f}{g}\right)(x)$ represents the quotient of two functions, $f(x)$ and $g(x)$, where $g(x) \neq 0$.
- The domain of a product or quotient of functions is the domain common to both $f(x)$ and $g(x)$. The domain of the quotient $\left(\frac{f}{g}\right)(x)$ is further restricted by excluding values where $g(x) = 0$.
- The range of a combined function can be determined using its graph.

Check Your Understanding

Practise

1. Determine $h(x) = f(x)g(x)$ and $k(x) = \dfrac{f(x)}{g(x)}$ for each pair of functions.

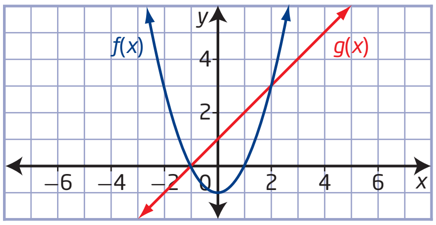
a) $f(x) = x + 7$ and $g(x) = x - 7$

b) $f(x) = 2x - 1$ and $g(x) = 3x + 4$

c) $f(x) = \sqrt{x + 5}$ and $g(x) = x + 2$

d) $f(x) = \sqrt{x - 1}$ and $g(x) = \sqrt{6 - x}$

2. Use the graphs of $f(x)$ and $g(x)$ to evaluate the following.



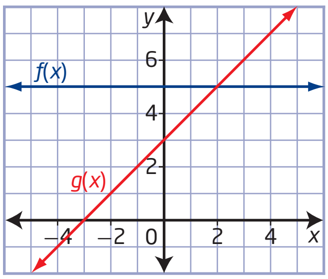
a) $(f \cdot g)(-2)$

b) $(f \cdot g)(1)$

c) $\left(\dfrac{f}{g}\right)(0)$

d) $\left(\dfrac{f}{g}\right)(1)$

3. Copy the graph. Add the sketch of the graph of each combined function to the same set of axes.



a) $h(x) = f(x)g(x)$

b) $h(x) = \dfrac{f(x)}{g(x)}$

4. For each pair of functions, $f(x)$ and $g(x)$,
- determine $h(x) = (f \cdot g)(x)$
- sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes
- state the domain and range of the combined function $h(x)$

a) $f(x) = x^2 + 5x + 6$ and $g(x) = x + 2$

b) $f(x) = x - 3$ and $g(x) = x^2 - 9$

c) $f(x) = \dfrac{1}{x + 1}$ and $g(x) = \dfrac{1}{x}$

5. Repeat #4 using $h(x) = \left(\dfrac{f}{g}\right)(x)$.

Apply

6. Given $f(x) = x + 2$, $g(x) = x - 3$, and $h(x) = x + 4$, determine each combined function.

a) $y = f(x)g(x)h(x)$

b) $y = \dfrac{f(x)g(x)}{h(x)}$

c) $y = \dfrac{f(x) + g(x)}{h(x)}$

d) $y = \dfrac{f(x)}{h(x)} \times \dfrac{g(x)}{h(x)}$

7. If $h(x) = f(x)g(x)$ and $f(x) = 2x + 5$, determine $g(x)$.

a) $h(x) = 6x + 15$

b) $h(x) = -2x^2 - 5x$

c) $h(x) = 2x\sqrt{x} + 5\sqrt{x}$

d) $h(x) = 10x^2 + 13x - 30$

8. If $h(x) = \dfrac{f(x)}{g(x)}$ and $f(x) = 3x - 1$, determine $g(x)$.

a) $h(x) = \dfrac{3x - 1}{x + 7}$

b) $h(x) = \dfrac{3x - 1}{\sqrt{x + 6}}$

c) $h(x) = 1.5x - 0.5$

d) $h(x) = \dfrac{1}{x + 9}$

9. Consider $f(x) = 2x + 5$ and $g(x) = \cos x$.

a) Graph $f(x)$ and $g(x)$ on the same set of axes and state the domain and range of each function.

b) Graph $y = f(x)g(x)$ and state the domain and range for the combined function.

10. Given $f(x)$ and $g(x)$, graph $y = \left(\dfrac{f}{g}\right)(x)$. State the domain and range of the combined function and any restrictions.

a) $f(x) = \tan x$ and $g(x) = \cos x$

b) $f(x) = \cos x$ and $g(x) = 0.8^x$

11. Let $f(x) = \sin x$ and $g(x) = \cos x$.

a) Write an expression as a quotient of functions that is equivalent to $\tan x$.

b) Write an expression as a product of functions that is equivalent to $1 - \cos^2 x$.

c) Use graphing technology to verify your answers to parts a) and b).

12. A fish farm plans to expand. The fish population, P, in hundreds of thousands, as a function of time, t, in years, can be modelled by the function $P(t) = 6(1.03)^t$. The farm biologists use the function $F(t) = 8 + 0.04t$, where F is the amount of food, in units, that can sustain the fish population for 1 year. One unit can sustain one fish for 1 year.

Fish farm at Sonora Island, British Columbia

a) Graph $P(t)$ and $F(t)$ on the same set of axes and describe the trends.

b) The amount of food per fish is calculated using $y = \frac{F(t)}{P(t)}$. Graph $y = \frac{F(t)}{P(t)}$ on a different set of axes. Identify a suitable window setting for your graph. Are there values that should not be considered?

c) At what time is the amount of food per fish a maximum?

d) The fish farm will no longer be viable when there is not enough food to sustain the population. When will this occur? Explain how you determined your result.

13. Let $f(x) = \sqrt{36 - x^2}$ and $g(x) = \sin x$.

a) Graph $f(x)$, $g(x)$, and $y = (f \cdot g)(x)$ on the same set of axes.

b) State the domain and range of the combined function.

c) Graph $y = \left(\frac{f}{g}\right)(x)$ and state its domain and range.

d) Explain how the domain and range for $y = \left(\frac{g}{f}\right)(x)$ differs from the domain and range in part c).

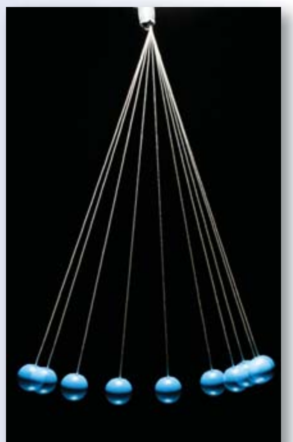
14. The motion of a damped harmonic oscillator can be modelled by a function of the form $d(t) = (A \sin kt) \times 0.4^{ct}$, where d represents the distance as a function of time, t, and A, k, and c are constants.

a) If $d(t) = f(t)g(t)$, identify the equations of the functions $f(t)$ and $g(t)$ and graph them on the same set of axes.

b) Graph $d(t)$ on the same set of axes.

Did You Know?

A damped harmonic oscillator is an object whose motion is cyclic with decreasing amplitude over time. Examples include a child on a swing after the initial push and a freely swinging pendulum.



Extend

15. The graph of $y = f(x)g(x)$, where $g(x)$ is a sinusoidal function, will oscillate between the graphs of $f(x)$ and $-f(x)$. When the amplitude of the wave is reduced, this is referred to as damping.
    a) Given the functions $f(x) = \frac{2}{x^2 + 1}$ and $g(x) = \sin(6x - 1)$, show that the above scenario occurs.
    b) Does the above scenario occur for $f(x) = \cos x$ and $g(x) = \sin(6x - 1)$?
16. The price, p, in dollars, set by a manufacturer for x tonnes of steel is $p(x) = 12x\left(\frac{x + 2}{x + 1}\right)$. Using the quotient of functions, determine whether the price per tonne decreases as the number of tonnes increases, algebraically and graphically.
17. A rectangle is inscribed in a circle of radius r. If the rectangle has length $2x$, write the area of the rectangle as the product of two functions.

Create Connections

C1 Is the product of functions commutative? Choose functions to represent $f(x)$ and $g(x)$ to explain whether $f(x)g(x) = g(x)f(x)$.

C2 Compare and contrast the properties of the domains of products of functions and quotients of functions.

C3 The volume, V, in cubic centimetres, of a square-based box is given by $V(x) = 4x^3 + 4x^2 - 39x + 36$.
   a) Write a combined function to represent the area, $A(x)$, of the base, if the side length of the base is $2x - 3$.
   b) Graph $A(x)$ and state its domain and range in this context.
   c) Determine the combined function $h(x) = \frac{V(x)}{A(x)}$. What does this represent in this context?
   d) Graph $h(x)$ and state its domain and range in this context.

Project Corner — Musical Presentation

Write lyrics to a song that demonstrate your understanding of a topic in Unit 4.

- You might examine the lyrics of a popular song to understand the structure (chorus and verse), line length, and rhymes.
- Choose a title for your song. Then, think of questions and answers that your title might suggest. Use a list of related words and phrases to help you write the lyrics.
- Finally, write your own melody or choose an existing melody to fit your lyrics to.

10.3

Composite Functions

Focus on...

- determining values of a composite function
- writing the equation of a composite function and explaining any restrictions
- sketching the graph of a composite function

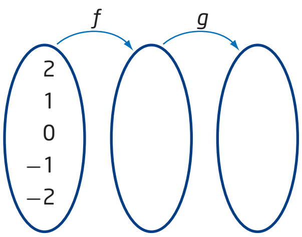
Approaching downtown Saskatoon, Saskatchewan

You have learned four ways of combining functions—adding, subtracting, multiplying, and dividing. Another type of combined function occurs any time a change in one quantity produces a change in another, which, in turn, produces a change in a third quantity. For example,

- the cost of travelling by car depends on the amount of gasoline consumed, and the amount of gasoline consumed depends on the number of kilometres driven
- the cost of an item on sale after taxes depends on the sale price, and the sale price of an item depends on the original price

Investigate Composition of Functions

1. Consider the functions $f(x) = 2x$ and $g(x) = x^2 + 2$. The output values of a function can become the input values for another function. Copy and complete the mapping diagram below.



2. Which of the following functions can be used to obtain the results from step 1 directly?

A $h(x) = 2x^3 + 4x$ **B** $h(x) = x^2 + 2x + 2$

C $h(x) = 2x^2 + 4$ **D** $h(x) = 4x^2 + 2$

3. Show, algebraically, how to create the equation you chose in step 2 from the original equations for f and g.

4. Suppose the output values of g become the input values for f.
   a) Create a mapping diagram to show this process.
   b) Write an equation that would give your results in one step.

Reflect and Respond

5. When working with two functions where one function is used as the input for the other function, does it make a difference which of the two functions is the input function? Explain.

6. Given the functions $f(x) = 4x + 2$ and $g(x) = -3x$ and using f as the input for g, list the steps you would use to determine a single equation to represent this situation.

7. Identify a real-life situation, different from the ones in the introduction to this section, where one function is used as the input for another function.

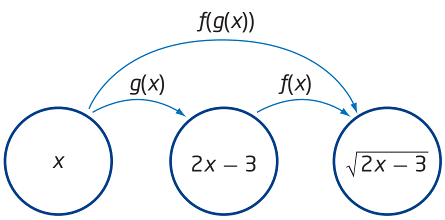
Link the Ideas

composite function

- the composition of $f(x)$ and $g(x)$ is defined as $f(g(x))$ and is formed when the equation of $g(x)$ is substituted into the equation of $f(x)$
- $f(g(x))$ exists only for those x in the domain of g for which $g(x)$ is in the domain of f
- $f(g(x))$ is read as "f of g of x" or "f at g of x" or "f composed with g"
- $(f \circ g)(x)$ is another way to write $f(g(x))$
- composition of functions must not be confused with multiplication, that is, $(f \circ g)(x)$ does not mean $(fg)(x)$

To compute $\sqrt{2(7) - 3}$ on many graphing calculators, the entire expression can be entered in one step. However, on some scientific calculators the expression must be entered in sequential steps. In this example, enter the expression $2(7) - 3$ and press the = button, which evaluates the calculation as 11. Then, press the √M button, which results in the square root of 11 or 3.3166…. The output of the expression $2(7) - 3$ is used as the input for the square root operation.

Composite functions are functions that are formed from two functions, $f(x)$ and $g(x)$, in which the output or result of one of the functions is used as the input for the other function. For example, if $f(x) = \sqrt{x}$ and $g(x) = 2x - 3$, then the composition of $f(x)$ and $g(x)$ is $f(g(x)) = \sqrt{2x - 3}$, as shown in the mapping diagram.



When composing functions, the order is important. $f(g(x))$ is not necessarily the same as $g(f(x))$. $f(g(x))$ means first substitute into g, and then substitute the result into f. On the other hand, $g(f(x))$ means first substitute into f, and then substitute the result into g.

Example 1

Evaluate a Composite Function

If $f(x) = 4x$, $g(x) = x + 6$, and $h(x) = x^2$, determine each value.

a) $f(g(3))$

b) $g(h(-2))$

c) $h(h(2))$

Solution

a) **Method 1: Determine the Value of the Inner Function and Then Substitute**

Evaluate the function inside the brackets for the indicated value of x. Then, substitute this value into the outer function.

Determine $g(3)$.

$g(x) = x + 6$

$g(3) = 3 + 6$

$g(3) = 9$

Substitute $g(3) = 9$ into $f(x)$.

$f(g(3)) = f(9)$ — Substitute 9 for $g(3)$.

$f(g(3)) = 4(9)$ — Evaluate $f(x) = 4x$ when x is 9.

$f(g(3)) = 36$

Method 2: Determine the Composite Function and Then Substitute

Determine the composite function first and then substitute.

$f(g(x)) = f(x + 6)$ — Substitute $x + 6$ for $g(x)$.

$f(g(x)) = 4(x + 6)$ — Substitute $x + 6$ into $f(x) = 4x$.

$f(g(x)) = 4x + 24$

Substitute $x = 3$ into $f(g(x))$.

$f(g(3)) = 4(3) + 24$

$f(g(3)) = 36$

When you evaluate composite functions, the result will not change if you compose first and then evaluate or evaluate first and then compose.

b) Determine $h(-2)$ and then $g(h(-2))$.

$h(x) = x^2$

$h(-2) = (-2)^2$

$h(-2) = 4$

Substitute $h(-2) = 4$ into $g(x)$.

$g(h(-2)) = g(4)$ — Substitute 4 for $h(-2)$.

$g(h(-2)) = 4 + 6$ — Evaluate $g(x) = x + 6$ when x is 4.

$g(h(-2)) = 10$

c) Determine $h(h(x))$ and then evaluate.

$h(h(x)) = h(x^2)$ Substitute x^2 for $h(x)$.

$h(h(x)) = (x^2)^2$ Substitute x^2 into $h(x) = x^2$.

$h(h(x)) = x^4$

Substitute $x = 2$ into $h(h(x))$.

$h(h(2)) = (2)^4$

$h(h(2)) = 16$

Your Turn

If $f(x) = |x|$ and $g(x) = x + 1$, determine $f(g(-11))$ using two methods.

Which method do you prefer? Why?

Example 2

Compose Functions With Restrictions

Consider $f(x) = \sqrt{x - 1}$ and $g(x) = x^2$.

a) Determine $(f \circ g)(x)$ and $(g \circ f)(x)$.

b) State the domain of $f(x)$, $g(x)$, $(f \circ g)(x)$, and $(g \circ f)(x)$.

Solution

a) Determine $(f \circ g)(x) = f(g(x))$.

$(f \circ g)(x) = f(x^2)$ Substitute x^2 for $g(x)$.

$(f \circ g)(x) = \sqrt{(x^2) - 1}$ Substitute x^2 into $f(x) = \sqrt{x - 1}$.

$(f \circ g)(x) = \sqrt{x^2 - 1}$

Determine $(g \circ f)(x) = g(f(x))$.

$(g \circ f)(x) = g(\sqrt{x - 1})$ Substitute $\sqrt{x - 1}$ for $f(x)$.

$(g \circ f)(x) = (\sqrt{x - 1})^2$ Substitute $\sqrt{x - 1}$ into $g(x) = x^2$.

$(g \circ f)(x) = x - 1$

Order does matter when composing functions. In this case, $(f \circ g)(x) \neq (g \circ f)(x)$.

b) The domain of $f(x)$ is $\{x \mid x \geq 1, x \in \mathrm{R}\}$.
The domain of $g(x)$ is $\{x \mid x \in \mathrm{R}\}$.

The domain of $(f \circ g)(x)$ is the set of all values of x in the domain of g for which $g(x)$ is in the domain of f. So, any restrictions on the inner function as well as the composite function must be taken into consideration.

- There are no restrictions on the domain of $g(x)$.
- The restriction on the domain of $(f \circ g)(x)$ is $x \leq -1$ or $x \geq 1$.

Combining these restrictions gives the domain of $(f \circ g)(x)$ as $\{x \mid x \leq -1 \text{ or } x \geq 1, x \in \mathrm{R}\}$.

The domain of $(g \circ f)(x)$ is the set of all values of x in the domain of f for which $f(x)$ is in the domain of g. So, any restrictions on the inner function and the composite function must be taken into consideration.

- The restriction on the domain of $f(x)$ is $x \geq 1$.
- There are no restrictions on the domain of $(g \circ f)(x)$.

Combining these restrictions gives the domain of $(g \circ f)(x)$ as $\{x \mid x \geq 1, x \in \mathrm{R}\}$.

Your Turn

Given the functions $f(x) = \sqrt{x - 1}$ and $g(x) = -x^2$, determine $(g \circ f)(x)$. Then, state the domain of $f(x)$, $g(x)$, and $(g \circ f)(x)$.

Example 3

Determine the Composition of Two Functions

Let $f(x) = x + 1$ and $g(x) = x^2$. Determine the equation of each composite function, graph it, and state its domain and range.

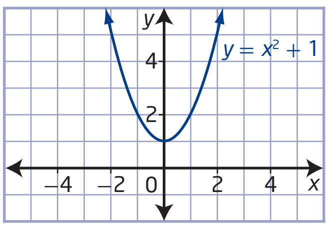
a) $y = f(g(x))$
b) $y = g(f(x))$
c) $y = f(f(x))$
d) $y = g(g(x))$

Solution

a) Determine $f(g(x))$.

$f(g(x)) = f(x^2)$
$f(g(x)) = (x^2) + 1$
$f(g(x)) = x^2 + 1$

The graph of the composite function $y = f(g(x))$ is a parabola that opens upward with vertex at (0, 1), domain of $\{x \mid x \in \mathrm{R}\}$, and range of $\{y \mid y \geq 1, y \in \mathrm{R}\}$.

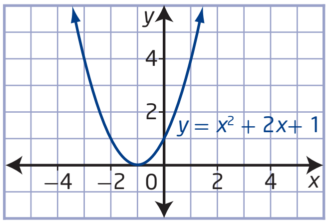


b) Determine $g(f(x))$.

$g(f(x)) = g(x + 1)$
$g(f(x)) = (x + 1)^2$
$g(f(x)) = x^2 + 2x + 1$

The graph of the composite function $y = g(f(x))$ is a parabola that opens upward with vertex at $(-1, 0)$, domain of $\{x \mid x \in R\}$, and range of $\{y \mid y \geq 0, y \in R\}$.

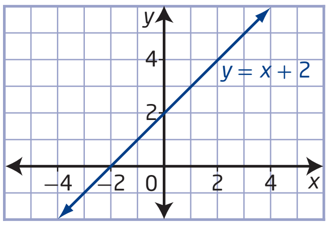
How do you know the coordinates of the vertex?



c) Determine $f(f(x))$.

$f(f(x)) = f(x + 1)$
$f(f(x)) = (x + 1) + 1$
$f(f(x)) = x + 2$

The graph of the composite function $y = f(f(x))$ represents a linear function. The domain and range of the function are both the set of real numbers.

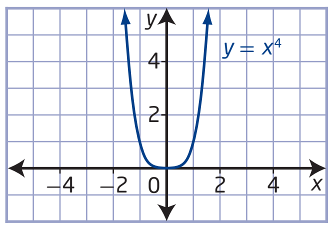


What are the slope and y-intercept of this line?

d) Determine $g(g(x))$.

$g(g(x)) = g(x^2)$
$g(g(x)) = (x^2)^2$
$g(g(x)) = x^4$

The graph of the composite function $y = g(g(x))$ is a quartic function that opens upward with domain of $\{x \mid x \in R\}$ and range of $\{y \mid y \geq 0, y \in R\}$.



Your Turn

Given $f(x) = |x|$ and $g(x) = x + 1$, determine the equations of $y = f(g(x))$ and $y = f(f(x))$, graph each composite function, and state the domain and range.

Example 4

Determine the Original Functions From a Composition

If $h(x) = f(g(x))$, determine $f(x)$ and $g(x)$.

a) $h(x) = (x - 2)^2 + (x - 2) + 1$

b) $h(x) = \sqrt{x^3 + 1}$

Solution

In this case, the inner function is $g(x)$ and the outer function is $f(x)$.

a) Look for a function that may be common to more than one term in $h(x)$. The same expression, $x - 2$, occurs in two terms. Let $g(x) = x - 2$. Then, work backward to determine $f(x)$.

$$h(x) = (x - 2)^2 + (x - 2) + 1$$
$$f(g(x)) = (g(x))^2 + (g(x)) + 1$$
$$f(x) = (x)^2 + (x) + 1$$

The two functions are $f(x) = x^2 + x + 1$ and $g(x) = x - 2$.

b) Let $g(x) = x^3 + 1$. Then, work backward to determine $f(x)$.

$$h(x) = \sqrt{x^3 + 1}$$
$$f(g(x)) = \sqrt{g(x)}$$
$$f(x) = \sqrt{x}$$

The two functions are $f(x) = \sqrt{x}$ and $g(x) = x^3 + 1$.

Is this the only solution? Explain.

Your Turn

If $h(x) = f(g(x))$, determine $f(x)$ and $g(x)$.

$$h(x) = \sqrt[3]{x} + \frac{3}{3 + \sqrt[3]{x}}$$

Example 5

Application of Composite Functions

A spherical weather balloon is being inflated. The balloon's radius, r, in feet, after t minutes is given by $r = \sqrt{t}$.

a) Express the volume of the balloon as a function of time, t.

b) After how many minutes will the volume be 4000 ft^3?

Cambridge Bay Upper Air, Nunavut

Did You Know?

The Global Climate Observing System (GCOS) Upper Air Network is a worldwide network of almost 170 stations that collect data for climate monitoring and research. Five of these stations are located in Canada, including Alert Upper Air Station, Nunavut; Cambridge Bay Upper Air Station, Nunavut; and Fort Smith Upper Air Station, Northwest Territories.

Solution

a) The formula for the volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$. Since r is a function of t, you can compose the two functions.

$$V(r) = \frac{4}{3}\pi r^3$$
$$V(r(t)) = \frac{4}{3}\pi(\sqrt{t})^3$$
$$V(r(t)) = \frac{4}{3}\pi\left(t^{\frac{1}{2}}\right)^3$$
$$V(r(t)) = \frac{4}{3}\pi t^{\frac{3}{2}}$$

b) To determine when the volume reaches 4000 ft³, substitute 4000 for V and solve for t.

$$V(r(t)) = \frac{4}{3}\pi t^{\frac{3}{2}}$$
$$4000 = \frac{4}{3}\pi t^{\frac{3}{2}}$$
$$\frac{3(4000)}{4\pi} = t^{\frac{3}{2}}$$
$$\left(\frac{3000}{\pi}\right)^{\frac{2}{3}} = \left(t^{\frac{3}{2}}\right)^{\frac{2}{3}}$$
$$t = \left(\frac{3000}{\pi}\right)^{\frac{2}{3}}$$
$$t = 96.972\ldots$$

After approximately 97 min, the volume will be 4000 ft³.

Your Turn

A spherical weather balloon is being blown up. The balloon's radius, r, in feet, after t minutes have elapsed is given by $r = \sqrt{t}$.

a) Express the surface area of the balloon as a function of time, t.

b) After how many minutes will the surface area be 180 ft²?

Key Ideas

- Two functions, $f(x)$ and $g(x)$, can be combined using composition to produce two new functions, $f(g(x))$ and $g(f(x))$.
- To evaluate a composite function, $f(g(x))$, at a specific value, substitute the value into the equation for $g(x)$ and then substitute the result into $f(x)$ and evaluate, or determine the composite function first and then evaluate for the value of x.
- To determine the equation of a composite function, substitute the second function into the first as read from left to right. To compose $f(g(x))$, substitute the equation of $g(x)$ into the equation of $f(x)$.
- The domain of $f(g(x))$ is the set of all values of x in the domain of g for which $g(x)$ is in the domain of f. Restrictions on the inner function as well as the composite function must be considered.

Check Your Understanding

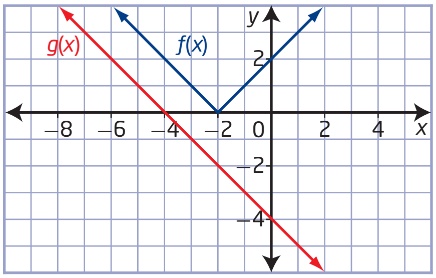
Practise

1. Given $f(2) = 3$, $f(3) = 4$, $f(5) = 0$, $g(2) = 5$, $g(3) = 2$, and $g(4) = -1$, evaluate the following.

a) $f(g(3))$ **b)** $f(g(2))$

c) $g(f(2))$ **d)** $g(f(3))$

2. Use the graphs of $f(x)$ and $g(x)$ to evaluate the following.



a) $f(g(-4))$ **b)** $f(g(0))$

c) $g(f(-2))$ **d)** $g(f(-3))$

3. If $f(x) = 2x + 8$ and $g(x) = 3x - 2$, determine each of the following.

a) $f(g(1))$ **b)** $f(g(-2))$

c) $g(f(-4))$ **d)** $g(f(1))$

4. If $f(x) = 3x + 4$ and $g(x) = x^2 - 1$, determine each of the following.

a) $f(g(a))$ **b)** $g(f(a))$

c) $f(g(x))$ **d)** $g(f(x))$

e) $f(f(x))$ **f)** $g(g(x))$

5. For each pair of functions, $f(x)$ and $g(x)$, determine $f(g(x))$ and $g(f(x))$.

a) $f(x) = x^2 + x$ and $g(x) = x^2 + x$

b) $f(x) = \sqrt{x^2 + 2}$ and $g(x) = x^2$

c) $f(x) = |x|$ and $g(x) = x^2$

6. Given $f(x) = \sqrt{x}$ and $g(x) = x - 1$, sketch the graph of each composite function. Then, determine the domain and range of each composite function.

a) $y = f(g(x))$

b) $y = g(f(x))$

7. If $h(x) = (f \circ g)(x)$, determine $g(x)$.

a) $h(x) = (2x - 5)^2$ and $f(x) = x^2$

b) $h(x) = (5x + 1)^2 - (5x + 1)$ and $f(x) = x^2 - x$

Apply

8. Ron and Christine are determining the composite function $(f \circ g)(x)$, where $f(x) = x^2 + x - 6$ and $g(x) = x^2 + 2$. Who is correct? Explain your reasoning.

Ron's Work

$$(f \circ g)(x) = f(g(x))$$
$$= (x^2 + 2)^2 + x - 6$$
$$= x^4 + 4x^2 + 4 + x - 6$$
$$= x^4 + 4x^2 + x - 2$$

Christine's Work

$$(f \circ g)(x) = f(g(x))$$
$$= (x^2 + 2)^2 + (x^2 + 2) - 6$$
$$= x^4 + 4x^2 + 4 + x^2 + 2 - 6$$
$$= x^4 + 5x^2$$

9. Let $j(x) = x^2$ and $k(x) = x^3$. Does $k(j(x)) = j(k(x))$ for all values of x? Explain.

10. If $s(x) = x^2 + 1$ and $t(x) = x - 3$, does $s(t(x)) = t(s(x))$ for all values of x? Explain.

11. A manufacturer of lawn chairs models the weekly production of chairs since 2009 by the function $C(t) = 100 + 35t$, where t is the time, in years, since 2009 and C is the number of chairs. The size of the workforce at the manufacturer's site is modelled by $W(C) = 3\sqrt{C}$.

a) Write the size of the workforce as a function of time.

b) State the domain and range of the new function in this context.

12. Tobias is shopping at a local sports store that is having a 25%-off sale on apparel. Where he lives, the federal tax adds 5% to the selling price.

a) Write the function, $s(p)$, that relates the regular price, p, to the sale price, s, both in dollars.

b) Write the function, $t(s)$, that relates the sale price, s, to the total cost including taxes, t, both in dollars.

c) Write a composite function that expresses the total cost in terms of the regular price. How much did Tobias pay for a jacket with a regular price of \$89.99?

13. Jordan is examining her car expenses. Her car uses gasoline at a rate of 6 L/100 km, and the average cost of a litre of gasoline where she lives is \$1.23.

a) Write the function, $g(d)$, that relates the distance, d, in kilometres, driven to the quantity, g, in litres, of gasoline used.

b) Write the function, $c(g)$, that relates the quantity, g, in litres, of gasoline used to the average cost, c, in dollars, of a litre of gasoline.

c) Write the composite function that expresses the cost of gasoline in terms of the distance driven. How much would it cost Jordan to drive 200 km in her car?

d) Write the composite function that expresses the distance driven in terms of the cost of gasoline. How far could Jordan drive her car on \$40?

14. Use the functions $f(x) = 3x$, $g(x) = x - 7$, and $h(x) = x^2$ to determine each of the following.

a) $(f \circ g \circ h)(x)$

b) $g(f(h(x)))$

c) $f(h(g(x)))$

d) $(h \circ g \circ f)(x)$

15. A Ferris wheel rotates such that the angle, θ, of rotation is given by $\theta = \frac{\pi t}{15}$, where t is the time, in seconds. A rider's height, h, in metres, above the ground can be modelled by $h(\theta) = 20 \sin \theta + 22$.

a) Write the equation of the rider's height in terms of time.

b) Graph $h(\theta)$ and $h(t)$ on separate sets of axes. Compare the periods of the graphs.

Did You Know?

The first Ferris wheel was designed for the 1893 World's Columbian Exposition in Chicago, Illinois, with a height of 80.4 m. It was built to rival the 324-m Eiffel Tower built for the 1889 Paris Exposition.

16. Environmental biologists measure the pollutants in a lake. The concentration, C, in parts per million (ppm), of pollutant can be modelled as a function of the population, P, of a nearby city, as $C(P) = 1.15P + 53.12$. The city's population, in ten thousands, can be modelled by the function $P(t) = 12.5(2)^{\frac{t}{10}}$, where t is time, in years.

a) Determine the equation of the concentration of pollutant as a function of time.

b) How long will it take for the concentration to be over 100 ppm? Show two different methods to solve this.

17. If $h(x) = f(g(x))$, determine $f(x)$ and $g(x)$.

a) $h(x) = 2x^2 - 1$

b) $h(x) = \frac{2}{3 - x^2}$

c) $h(x) = |x^2 - 4x + 5|$

18. Consider $f(x) = 1 - x$ and $g(x) = \frac{x}{1 - x}$, $x \neq 1$.

a) Show that $g(f(x)) = \frac{1}{g(x)}$.

b) Does $f(g(x)) = \frac{1}{f(x)}$?

19. According to Einstein's special theory of relativity, the mass, m, of a particle moving at velocity v is given by $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$, where m_0 is the particle's mass at rest and c is the velocity of light. Suppose that velocity, v, in miles per hour, is given as $v = t^3$.

a) Express the mass as a function of time.

b) Determine the particle's mass at time $t = \sqrt[3]{\frac{c}{2}}$ hours.

Extend

20. In general, two functions $f(x)$ and $g(x)$ are inverses of each other if and only if $f(g(x)) = x$ and $g(f(x)) = x$. Verify that the pairs of functions are inverses of each other.

a) $f(x) = 5x + 10$ and $g(x) = \frac{1}{5}x - 2$

b) $f(x) = \frac{x - 1}{2}$ and $g(x) = 2x + 1$

c) $f(x) = \sqrt[3]{x + 1}$ and $g(x) = x^3 - 1$

d) $f(x) = 5^x$ and $g(x) = \log_5 x$

21. Consider $f(x) = \log x$ and $g(x) = \sin x$.

a) What is the domain of $f(x)$?

b) Determine $f(g(x))$.

c) Use a graphing calculator to graph $y = f(g(x))$. Work in radians.

d) State the domain and range of $y = f(g(x))$.

22. If $f(x) = \frac{1}{1 + x}$ and $g(x) = \frac{1}{2 + x}$, determine $f(g(x))$.

23. Let $f_1(x) = x$, $f_2(x) = \frac{1}{x}$, $f_3(x) = 1 - x$, $f_4(x) = \frac{x}{x - 1}$, $f_5(x) = \frac{1}{1 - x}$, and $f_6(x) = \frac{x - 1}{x}$.

a) Determine the following.

i) $f_2(f_3(x))$

ii) $(f_3 \circ f_5)(x)$

iii) $f_1(f_2(x))$

iv) $f_2(f_1(x))$

b) $f_6^{-1}(x)$ is the same as which function listed in part a)?

Create Connections

C1 Does $f(g(x))$ mean the same as $(f \cdot g)(x)$? Explain using examples.

C2 Let $f = \{(1, 5), (2, 6), (3, 7)\}$ and $g = \{(5, 10), (6, 11), (7, 0)\}$. Explain how each equation is true.

a) $g(f(1)) = 10$

b) $g(f(3)) = 0$

C3 Suppose that $f(x) = 4 - 3x$ and $g(x) = \frac{4 - x}{3}$. Does $g(f(x)) = f(g(x))$ for all x? Explain.

C4 MINI LAB

Step 1 Consider $f(x) = 2x + 3$.

a) Determine $f(x + h)$.

b) Determine $\frac{f(x + h) - f(x)}{h}$.

Step 2 Repeat step 1 with $f(x) = -3x - 5$.

Step 3 Predict what $\frac{f(x + h) - f(x)}{h}$ will be for $f(x) = \frac{3}{4}x - 5$. How are each of the values you found related to the functions?

Chapter 10 Review

10.1 Sums and Differences of Functions, pages 474–487

1. Given $f(x) = 3x - 1$ and $g(x) = 2x + 7$, determine

a) $(f + g)(4)$ **b)** $(f + g)(-1)$

c) $(f - g)(3)$ **d)** $(g - f)(-5)$

2. Consider the functions $g(x) = x + 2$ and $h(x) = x^2 - 4$.

a) Determine the equation and sketch the graph of each combined function. Then, state the domain and range.

i) $f(x) = g(x) + h(x)$

ii) $f(x) = h(x) - g(x)$

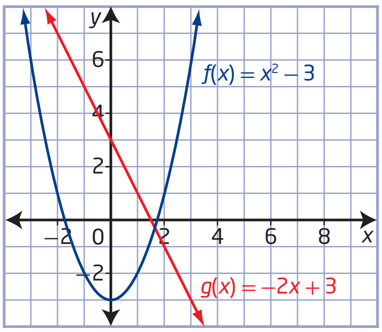
iii) $f(x) = g(x) - h(x)$

b) What is the value of $f(2)$ for each combined function in part a)?

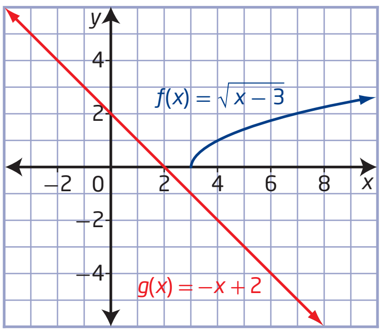
3. For each graph of $f(x)$ and $g(x)$,

- determine the equation and graph of $y = (f + g)(x)$ and state its domain and range
- determine the equation and graph of $y = (f - g)(x)$ and state its domain and range

a)



b)



4. Let $f(x) = \dfrac{1}{x - 1}$ and $g(x) = \sqrt{x}$. Determine the equation of each combined function and state its domain and range.

a) $(f + g)(x)$

b) $(f - g)(x)$

5. A biologist has been recording the births and deaths of a rodent population on several sections of farmland for the past 5 years. Suppose the function $b(x) = -4x + 78$ models the number of births and the function $d(x) = -6x + 84$ models the number of deaths, where x is the time, in years. The net change in population, P, is equal to the number of births minus the number of deaths.

a) Write an expression that reflects the net change in population at any given time.

b) Assuming that the rates continue, predict how the population of rodents will behave over the next 5 years.

c) At what point in time does the population start to increase? Explain.

10.2 Products and Quotients of Functions, pages 488–498

6. Consider the functions $g(x) = x + 2$ and $h(x) = x^2 - 4$. Determine the equation and sketch the graph of each combined function $f(x)$. Then, state the domain and range and identify any asymptotes.

a) $f(x) = g(x)h(x)$

b) $f(x) = \dfrac{h(x)}{g(x)}$

c) $f(x) = \dfrac{g(x)}{h(x)}$

7. Determine the value of $f(-2)$ for each combined function in #6.

8. Given $g(x) = \frac{1}{x + 4}$ and $h(x) = \frac{1}{x^2 - 16}$, determine the equation of each combined function and state its domain and range.

a) $f(x) = g(x)h(x)$

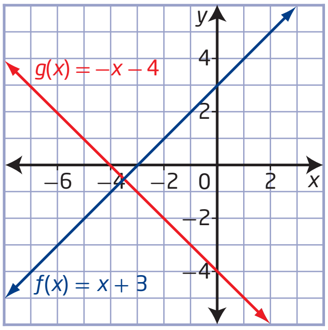
b) $f(x) = \frac{g(x)}{h(x)}$

c) $f(x) = \frac{h(x)}{g(x)}$

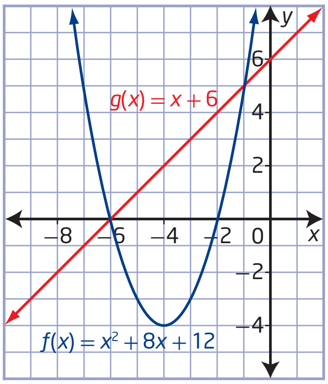
9. For each graph of $f(x)$ and $g(x)$,

- determine the equation and graph of $y = (f \cdot g)(x)$ and state its domain and range
- determine the equation and graph of $y = \left(\frac{f}{g}\right)(x)$ and state its domain and range

a)



b)



10.3 Composite Functions, pages 499–509

10. Given $f(x) = x^2$ and $g(x) = x + 1$, determine the following.

a) $f(g(-2))$

b) $g(f(-2))$

11. For $f(x) = 2x^2$ and $g(x) = \frac{4}{x}$, determine the following and state any restrictions.

a) $f(g(x))$

b) $g(f(x))$

c) $g(f(-2))$

12. Consider $f(x) = -\frac{2}{x}$ and $g(x) = \sqrt{x}$.

a) Determine $y = f(g(x))$.

b) State the domain and range of $y = f(g(x))$.

13. If $f(x) = 2x - 5$ and $g(x) = x + 6$, determine $y = (f \circ g)(x)$. Then, sketch the graphs of the three functions.

14. The temperature of Earth's crust is a linear function of the depth below the surface. An equation expressing this relationship is $T = 0.01d + 20$, where T is the temperature, in degrees Celsius, and d is the depth, in metres. If you go down a vertical shaft below ground in an elevator at a rate of 5 m/s, express the temperature as a function of time, t, in seconds, of travel.

15. While shopping for a tablet computer, Jolene learns of a 1-day sale of 25% off. In addition, she has a coupon for $10 off.

a) Let x represent the current price of the tablet. Express the price, d, of the tablet after the discount and the price, c, of the tablet after the coupon as functions of the current price.

b) Determine $c(d(x))$ and explain what this function represents.

c) Determine $d(c(x))$ and explain what this function represents.

d) If the tablet costs $400, which method results in the lower sale price? Explain your thinking.

Chapter 10 Practice Test

Multiple Choice

For #1 to #5, choose the best answer.

1. Let $f(x) = (x + 3)^2$ and $g(x) = x + 4$. Which function represents the combined function $h(x) = f(x) + g(x)$?

A $h(x) = x^2 + 7x + 7$

B $h(x) = x^2 + 7x + 13$

C $h(x) = x^2 + x + 13$

D $h(x) = x^2 + 2x + 7$

2. If $f(x) = x + 8$ and $g(x) = 2x^2 - 128$, what is the domain of $y = \frac{g(x)}{f(x)}$?

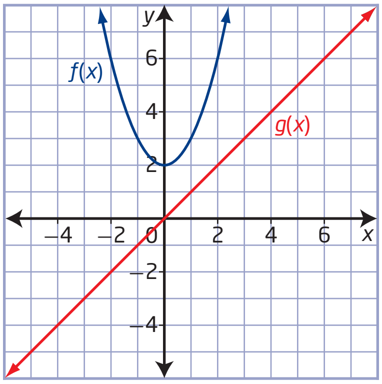
A $\{x \mid x \in \mathrm{R}\}$

B $\{x \mid x \in \mathrm{I}\}$

C $\{x \mid x \neq 8, x \in \mathrm{R}\}$

D $\{x \mid x \neq -8, x \in \mathrm{R}\}$

3. The graphs of two functions are shown.



Which is true for $x \in \mathrm{R}$?

A $g(x) - f(x) < 0$

B $\frac{f(x)}{g(x)} > 1, x \neq 0$

C $f(x) < g(x)$

D $g(x) + f(x) < 0$

4. Given $f(x) = 5 - x$ and $g(x) = 2\sqrt{3x}$, what is the value of $f(g(3))$?

A $5 - 2\sqrt{6}$

B $2\sqrt{15 - 3x^2}$

C -1

D 1

5. Which function represents $y = f(g(x))$, if $f(x) = x + 5$ and $g(x) = x^2$?

A $y = x^2 + 5$

B $y = x^2 + 25$

C $y = x^2 + x + 5$

D $y = x^2 + 10x + 25$

Short Answer

6. Given $f(x) = \sin x$ and $g(x) = 2x^2$, determine each combined function.

a) $h(x) = (f + g)(x)$

b) $h(x) = (f - g)(x)$

c) $h(x) = (f \cdot g)(x)$

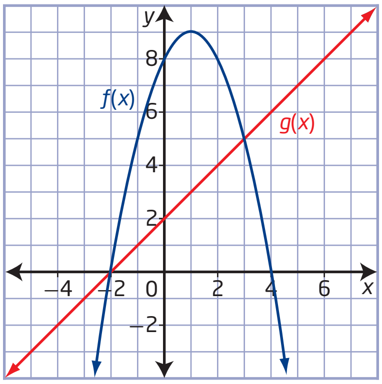
d) $h(x) = \left(\frac{f}{g}\right)(x)$

7. Copy and complete the table by determining the missing terms.

	$g(x)$	$f(x)$	$(f + g)(x)$	$(f \circ g)(x)$
a)	$x - 8$	$\sqrt{x}$		
b)	$x + 3$	$4x$		
c)		$\sqrt{x - 4}$		$\sqrt{x^2 - 4}$
d)	$\frac{1}{x}$			x

8. Determine the product of $g(x) = \frac{1}{1 + x}$ and $h(x) = \frac{1}{3 + 2x}$. Then, state the domain of the combined function.

9. Use the graphs of $f(x)$ and $g(x)$ to sketch the graph of each combined function.



a) $y = (f - g)(x)$

b) $y = \left(\frac{f}{g}\right)(x)$

10. For each of the following pairs of functions, determine $g(f(x))$ and state its domain and range.

a) $f(x) = 3 - x$ and $g(x) = |x + 3|$

b) $f(x) = 4^x$ and $g(x) = x + 1$

c) $f(x) = x^4$ and $g(x) = \sqrt{x}$

11. Becky has \$200 deducted from every paycheque for her retirement. This can be done before or after federal income tax is assessed. Suppose her federal income tax rate is 28%.

a) Let x represent Becky's earnings per pay period. Represent her income, r, after the retirement deduction and her income, t, after federal taxes as functions of her earnings per pay period.

b) Determine $t(r(x))$. What does this represent?

c) If Becky earns \$2700 every pay period, calculate her net income using the composite function from part b).

d) Calculate Becky's net income using $r(t(x))$.

e) Explain the differences in net income.

12. A pendulum is released and allowed to swing back and forth according to the equation $x(t) = (10 \cos 2t)(0.95^t)$, where x is the horizontal displacement from the resting position, in centimetres, as a function of time, t, in seconds.

a) Graph the function.

b) The equation is the product of two functions. Identify each function and explain which is responsible for
- the periodic motion
- the exponential decay of the amplitude

Extended Response

13. Given $f(x) = 2x^2 + 11x - 21$ and $g(x) = 2x - 3$, determine the equation and sketch the graph of each combined function.

a) $y = f(x) - g(x)$

b) $y = f(x) + g(x)$

c) $y = \frac{f(x)}{g(x)}$

d) $y = f(g(x))$

14. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 50 cm/s.

a) Write an equation that represents the area of the circle as a function of time. State the type of combined function you wrote.

b) Graph the function.

c) What is the area of the circle after 5 s?

d) Is it reasonable to calculate the area of the circle after 30 s? Explain.